Portfolio Diversification

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Total variance of portfolio returns can be reduced by combining different assets whose returns are not correlated. Portfolio return will be the weighted average of individual asset returns but portfolio variance will be less than the weighted average of individual asset return variances. This paper will develop the mathematics to calculate expected portfolio return and return variance, and will work through a hypothetical case to demonstrate it's application.

Legend of Symbols

N	=	Number of assets in portfolio
d_i	=	Dollar return on asset i
r_p	=	Portfolio percentage return
v_i	=	Dollars invested in asset i
w_i	=	Dollar weight of asset i in portfolio
z_i	=	Normally-distributed random number (mean = 0, variance = 1)
μ_i	=	Return on asset i - mean
σ_i	=	Return on asset i - standard deviation
$\rho_{i,j}$	=	Correlation between asset i and asset j

Portfolio Return

The return on any asset is the sum of expected return plus unexpected return. Expected return is related to return mean, unexpected return is related to return variance. If we assume that asset returns are normally-distributed, the dollar return on any asset i is given by the following equation...

$$d_i = v_i \times (\mu_i + \sigma_i z_i) \tag{1}$$

Portfolio percentage return is dollar return divided by portfolio dollar value. For a portfolio of assets, the percentage return equation is...

$$r_p = \sum_{i=1}^{N} v_i \times (\mu_i + \sigma_i z_i) \div \sum_{i=1}^{N} v_i \tag{2}$$

We will define w_i as the dollar weight of asset i in the portfolio. The equation for dollar weight of asset i is...

$$w_i = v_i \div \sum_{j=1}^{N} v_j \tag{3}$$

We can now rewrite portfolio percentage return equation (2) as...

$$r_p = \sum_{i=1}^{N} w_i \times (\mu_i + \sigma_i z_i) \tag{4}$$

It is obvious from equation (4) that portfolio percentage return is the weighted average return of the individual assets in the portfolio. If the assets in the portfolio are not perfection correlated then portfolio return variance is less than the weighted average variance of the individual assets in the portfolio. Diverification is the process of combining assets such that portfolio return equals the weighted average but portfolio risk (variance) is less than the weighted average.

Portfolio Return Moments

We will assume that asset returns are normally-distributed. The first moment of the return distribution is the expected portfolio return (i.e. the mean). The equation for the first moment is the expected value of r_p or...

$$\begin{aligned}\mathbb{E}\Big[r_p\Big] &= \mathbb{E}\Big[\sum_{i=1}^{N} w_i \times (\mu_i + \sigma_i z_i)\Big]\\ &= \mathbb{E}\Big[\sum_{i=1}^{N} w_i \mu_i\Big] + \mathbb{E}\Big[\sum_{i=1}^{N} w_i \sigma_i z_i)\Big]\\ &= \sum_{i=1}^{N} w_i \mu_i \end{aligned} \tag{5}$$

As the number of assets in the portfolio grows it is often times more efficient to work with matrices. Expected return is the product of the column vector of asset weights and the row vector of asset expected returns. Matrix multiplication is easily done in software packages such as Excel (MMult function). In matrix notation the expected portfolio return becomes (assuming a three asset portfolio)...

$$\sum_{i=1}^{N} w_i \mu_i = \begin{vmatrix} w_1 & w_2 & w_3 \end{vmatrix} \times \begin{vmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{vmatrix} \tag{6}$$

The second moment of the portfolio return distribution is related to portfolio return variance. The equation for the second moment is the weighted sum of the squares of portfolio returns. The equation for the second moment is the expected value of r_p^2 or...

$$\begin{aligned}\mathbb{E}\Big[r_p^2\Big] &= \mathbb{E}\Big[\sum_{i=1}^{N} w_i \times (\mu_i + \sigma_i z_i) \sum_{j=1}^{N} w_j \times (\mu_j + \sigma_j z_j)\Big]\\ &= \mathbb{E}\Big[\Big[\sum_{i=1}^{N} w_i \mu_i + \sum_{i=1}^{N} w_i \sigma_i z_i\Big]\Big[\sum_{j=1}^{N} w_j \mu_j + \sum_{j=1}^{N} w_i \sigma_j z_j\Big]\Big]\\ &= \mathbb{E}\Big[r_p\Big]^2 + \sum_{i=1}^{N}\sum_{j=1}^{N} w_i\, w_j\, cov(i,j) \end{aligned} \tag{7}$$

The variance of r_p is the second moment (equation (7)) minus the square of the first moment (equation (5)). The equation for the variance of portfolio return is...

$$\begin{aligned} Variance &= \mathbb{E}\Big[r_p^2\Big] - \mathbb{E}\Big[r_p\Big]^2\\ &= \sum_{i=1}^{N}\sum_{j=1}^{N} w_i\, w_j\, cov(i,j) \end{aligned} \tag{8}$$

Portfolio return variance is the product of the column vector of asset weights, the covariance matrix and the row vector of asset weights. In matrix notation (assuming a three asset portfolio)...

$$\sum_{i=1}^{N}\sum_{j=1}^{N} w_i\, w_j\, cov(i,j) = \begin{vmatrix} w_1 & w_2 & w_3 \end{vmatrix} \times \begin{vmatrix} \sigma_1^2 & \sigma_1\,\sigma 2\,\rho 12 & \sigma_1\,\sigma 3\,\rho 13 \\ \sigma_2\,\sigma 1\,\rho 21 & \sigma_2^2 & \sigma_2\,\sigma 3\,\rho 23 \\ \sigma_3\,\sigma 1\,\rho 31 & \sigma_3\,\sigma 2\,\rho 32 & \sigma_3^2 \end{vmatrix} \times \begin{vmatrix} w_1 \\ w_2 \\ w_3 \end{vmatrix} \tag{9}$$

A Hypothetical Case

Assume that we have a three asset portfolio for which we want to calculate the expected return and variance. Because the asset returns are not perfectly correlated portfolio return variance should be less than weighted average asset variance.

Asset weight, expected return and volatilities:

Asset	Weight	Mean	Volatility
1	15%	10%	15%
2	55%	20%	30%
3	30%	30%	40%

Asset correlation matrix:

Asset	1	2	3
1	1.00	0.18	0.24
2		1.00	0.48
3			1.00

Expected portfolio return calculation:

$$\begin{vmatrix} 0.15 & 0.55 & 0.30 \end{vmatrix} \times \begin{vmatrix} 0.10 \\ 0.20 \\ 0.30 \end{vmatrix} = 0.2150$$

Portfolio variance calculation:

$$\begin{vmatrix} 0.15 & 0.55 & 0.30 \end{vmatrix} \times \begin{vmatrix} 0.0225 & 0.0081 & 0.0144 \\ 0.0081 & 0.0900 & 0.0576 \\ 0.0144 & 0.0576 & 0.1600 \end{vmatrix} \times \begin{vmatrix} 0.15 \\ 0.55 \\ 0.30 \end{vmatrix} = 0.0638$$

Expected return of 21.50% with a standard deviation of 25.25% (square root of 0.0638). The weighted average volatility (i.e. no diversification benefits) is 30.75%.